Separation of Variables 分离变量 (p406)

Consider a differential equation that can be written in the form

$$M(x)+\frac{N(y)dy}{dx}=0$$

$M(x)$ is a continuous function of x alone
$N(y)$ is a continuous function of y alone.

$$M(x)+\frac{N(y)dy}{dx}=0 \Rightarrow M(x)dx=-N(y)dy$$

Such equations are said to be **separable**, and the solution procedure is called **separation of variables(分离变量)**.

Original Differential Equation	*Rewritten with Variables Separated*
$x^2+3y\frac{dy}{dx}=0$	$3ydy=-x^2dx$
$(\sin x)y'=\cos x$	$dy=\cot x\,dx$
$\frac{xy'}{e^y+1}=2$	$\frac{dy}{e^y+1}=\frac{2dx}{x}$

Example: Find the general solution of

$$\frac{(x^2+4)dy}{dx}=xy$$

Solution: To begin, note that $y=0$ is a solution. To find other solutions, assume that $y\neq 0$ and separate variables as shown.

$$\Rightarrow (x^2+4)dy=xydx$$

$$\frac{dy}{y}=\frac{xdx}{x^2+4}$$ Separate variables

$$\int\frac{dy}{y}=\int\frac{xdx}{x^2+4}$$

$$u=x^2+4\Rightarrow du=2xdx\quad(u>0)$$

$$\int\frac{dy}{y}=\int\frac{(1/2)du}{u}$$

$$\ln|y|=\frac{1}{2}\ln(u)+C_1=\frac{1}{2}\ln(x^2+4)+C_1$$

$$|y|=e^{C_1}\sqrt{x^2+4}\Rightarrow y=\pm e^{C_1}\sqrt{x^2+4}$$

$$y=C\sqrt{x^2+4}$$ Let $C=\pm e^{C_1}$ ($wheter\ C=0?$)

REMARK: check solutions throughout this chapter.

$$y=C\sqrt{x^2+4}$$

$$\Rightarrow\frac{dy}{dx}=C\frac{d}{dx}\sqrt{x^2+4}=C\left[\frac{1}{2}(x^2+4)^{\frac{-1}{2}}\cdot 2x\right]$$

$$\Rightarrow\frac{dy}{dx}=\frac{x\cdot C}{\sqrt{x^2+4}}=\frac{x\cdot C\sqrt{x^2+4}}{x^2+4}=\frac{xy}{x^2+4}$$

$$\Rightarrow\frac{(x^2+4)dy}{dx}=xy$$

结论：可分离变量的微分方程，经常需要检查 y=0 是否是一个解。

In some cases, it is not feasible to write the general solution in the explicit form $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can be used to verify this solution.

Example: Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}(y^2-1)dy = 0$$

Solution: $xy\,dx = -e^{-x^2}(y^2-1)dy$

$$\frac{xdx}{-e^{-x^2}} = \frac{(y^2-1)dy}{y} \Rightarrow -xe^{x^2}dx = (y-\frac{1}{y})dy$$

$$-\int xe^{x^2}dx = \int\left(y-\frac{1}{y}\right)dy \Rightarrow \frac{-1}{2}\int e^{x^2}dx^2 = \int\left(y-\frac{1}{y}\right)dy$$

$$\frac{-1}{2}e^{x^2} = \frac{y^2}{2} - \ln|y| + c$$

From the initial condition $y(0) = 1$, you have

$$\frac{-1}{2}e^0 = \frac{1}{2} - \ln 1 + c \Rightarrow \frac{-1}{2} = \frac{1}{2} - 0 + c$$

$$c = -1$$

So, the particular solution has the implicit form

$$\frac{-1}{2}e^{x^2} = \frac{y^2}{2} - \ln|y| - 1 \Rightarrow y^2 - \ln y^2 + e^{x^2} = 2$$

Please notice that $y = 0$ can also satisfy the differential equation but not meet the initial condition $y(0) = 1$.

Example: Find the equation of the curve that passes through the point (1,3) and has a slope of y/x^2 at any point (x, y).

Solution: $y' = y/x^2$

$$\frac{dy}{dx} = \frac{y}{x^2}$$

$$\frac{dy}{y} = \frac{dx}{x^2} \quad (y \neq 0)$$

$$\int\frac{dy}{y} = \int\frac{dx}{x^2}$$

$$\ln|y| = \frac{x^{-1}}{-1} + C = \frac{-1}{x} + C$$

the curve that passes through the point (1,3), so that y>0.

$$\ln y = \frac{-1}{x} + C$$

$$\ln 3 = \frac{-1}{1} + C$$

$$C = \ln 3 + 1$$

The particular solution is

$$\ln y = \frac{-1}{x} + \ln 3 + 1$$

or $y = 3e^{(x-1)/x}$

为什么$x > 0$, Ron Larson Cal 10th P416. 不明白?

A common problem in electrostatics, thermodynamics, and hydrodynamics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves.

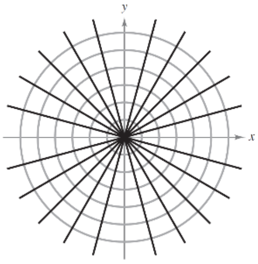
The right figure shows a family of circles

$x^2+y^2=C$ Family of circles

each of which intersects the lines in the family

$y=kx$ Family of lines

at right angles. Two such families of curves are said to be **mutually orthogonal**, and each curve in one of the families is called an **orthogonal trajectory** of the other family.



equipotential curves

Example: Describe the orthogonal trajectories for the family of curves given by

$$y=C/x \quad C\neq 0$$

Solution

$y=C/x \Rightarrow xy=C$ original family curves

$$y+x\frac{dy}{dx}=0$$

$$x\frac{dy}{dx}=-y$$

$\dfrac{dy}{dx}=-\dfrac{y}{x}$ slop of given family

Because dy/dx represents the slope of the given family of (x,y) curves at it follows that the orthogonal family has the negative reciprocal slope $x/$y. So,

$\dfrac{dy}{dx}=\dfrac{x}{y}$ slop of orthogonal family

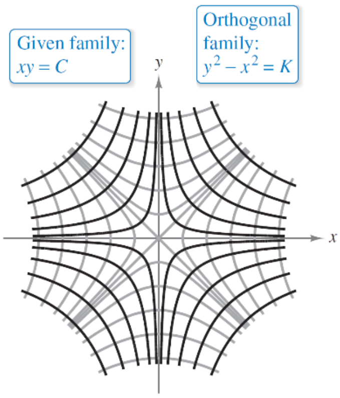
Solve this differential equation

$$ydy=xdx$$

$$\int ydy=\int xdx$$

$$\frac{y^2}{2}=\frac{x^2}{2}+C_1$$

$y^2-x^2=K$ *othogonal family*



Exercise: Calculus 10th, Ron Larson,
P421: 7,8,11,12,13,14,21,22,23,24,25,26,43,44